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The influence of drift flow turbulence on surface gravity wave propagation

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The theory of surface gravity waves scattering at vortex flows in the ocean is developed in this paper. A scattering amplitude is found in the Born approximation as a function of vorticity which appears very convenient for investigation of scattering at simple localized flows. It is shown that the wave scattering cross-section is determined by the vertical component of vorticity. For a random (turbulent) vortex field the scattering cross-section per unit volume is determined by a vorticity correlation function. The damping of the coherent wave component and the angular spectrum widening are calculated for multiple scattering by vortex turbulence of drift flows. The spectrum angular width evolution for waves scattered at self-similar vortices of the logarithmic boundary layer is determined only by its dynamical speed and the wave vector. The latter result may be used for a remote sensing of oceanic turbulent drift flows based on observations of surface waves.

1. Introduction

Vortex flows of various scales are among most important natural factors influencing surface gravity waves in the ocean. Flow fields of synoptic and meso-scale vortices produce considerable refraction and other effects on waves propagating from any generation area (Phillips 1984; Hayes 1980; Sheres & Kenyon 1990). Small-scale vortices generated by upper-layer convection, shear flow instability or wave breaking also play an important role in surface wave dynamics (see, for example, Monin & Ozmidov 1981; Longuet-Higgins 1992).

But some of physical mechanisms for the influence of subsurface vortices on gravity wave propagation are still unclear. For example, the effect of wave damping in turbulence has been treated before in terms of turbulent viscosity (see Kononkova 1969; Kitaigorodskii & Lumley 1983), but the estimations of a wave decrement obtained in such a fashion are rather discrepant: they lead to very different values and fail to reveal physical features of the processes taking place.

Here we wish to consider the phenomenon of surface wave scattering at subsurface vortices. First we investigate wave scattering at a localized vortical area. Then scattering by vortex turbulence is considered with special attention to the turbulence of wind drift flow.

The effect of surface wave scattering by turbulence was first investigated by Phillips (1959). Some results concerning gravity wave scattering by spatially homogeneous and horizontally isotropic turbulence were obtained by Raevsky (1983) and Sazontov & Shagalov (1985). But it is obvious that more realistic models should take into account the vertical inhomogeneity of subsurface turbulence.

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Turbulence in an ocean can be caused by various factors: internal waves, wind wave breaking and so on (see Phillips 1977; Monin & Ozmidov 1981). We wish to emphasize however that subsurface turbulence caused by wind drift flows is one of the most important forms of turbulence that influences surface waves. Note that the interaction with drift flow turbulence is an inherent effect for wind waves, whereas turbulence of any other origin is an incidental factor whose random appearance is not caused by wind. Besides, the effect of drift turbulence on the damping and scattering of swell propagating through a storm area is also of considerable interest.

A wind drift flow appears near a sea surface mainly under the influence of a tangential wind stress. The features of this flow are quite similar to those of a turbulent boundary layer on a solid surface (see Wu 1975; Jones & Kenney 1977; Lin & Gad-el-Hak 1984). In particular, at a depth greater than the viscous-layer thickness and up to the external turbulence scale L_t (which is usually determined by stratification or by the Ekman scale) the logarithmic boundary layer (LBL) approximation is valid, which has self-similar properties (see, for example, Phillips 1977). Here we consider surface gravity wave scattering by vertically inhomogeneous turbulence of the LBL. This problem has not been investigated before but it seems to be important for wind wave and swell dynamics.

2. Basic equations

We use equations for an ideal incompressible fluid and boundary conditions at the free surface $z = \zeta(x, y)$ and at the bottom $z = -H$:

$$\frac{\partial \boldsymbol{V}}{\partial t}+\nabla\left(\frac{V^2}{2}+\frac{p}{\rho}+gz\right) = \boldsymbol{V}\times[\nabla\times \boldsymbol{V}], \tag{1a}$$

$$\nabla\cdot \boldsymbol{V} = 0, \tag{1b}$$

$$\frac{\partial \zeta}{\partial t}+(\boldsymbol{V}_\perp\cdot\nabla_\perp)\,\zeta = V_z(z=\zeta),\quad p(z=\zeta)=0,\quad V_z(z=-H)=0. \tag{1c–e}$$

Here $$\boldsymbol{V} = (V_x, V_y, V_z),\quad \boldsymbol{V}_\perp = (V_x, V_y, 0),\quad \nabla_\perp = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},0\right).$$

To investigate waves of small amplitude propagating on a background flow we consider the fluid motion as a superposition of an undisturbed vortex flow $\boldsymbol{U}(\boldsymbol{r})$ (in particular, it can be turbulence with given statistical properties) and a perturbation caused by a surface wave:

$$\boldsymbol{V} = \boldsymbol{U}+\boldsymbol{v},\quad \zeta = h+\eta,\quad p = P-\rho g z+\tilde{p}, \tag{2}$$

where $\boldsymbol{v}(\boldsymbol{r})$, η and $\tilde{p}$ are wave fields of the velocity, surface displacement and pressure.

The following approximations will be used below.

(i) *Small wave amplitude*. We will develop below a linear surface wave theory and so we can neglect terms of the form $\boldsymbol{v}\cdot\boldsymbol{v}$. Linearized equations for infinitesimal wave perturbations on a vortex flow background have the form:

$$\frac{\partial \boldsymbol{v}}{\partial t}+\nabla\left(\frac{\tilde{p}}{\rho}\right)+(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{U}+(\boldsymbol{U}\cdot\nabla)\,\boldsymbol{v} = 0, \tag{3a}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{3b}$$

$$\frac{\partial \eta}{\partial t}-v_z(z=0)+(\boldsymbol{U}_\perp\cdot\nabla_\perp)\,\eta-\frac{\partial U_z}{\partial z}\eta = -(\boldsymbol{v}_\perp\cdot\nabla_\perp)\,h+\frac{\partial v_z}{\partial z}h, \tag{3c}$$

$$\tilde{p}(z=0)-\rho g h = -\frac{\partial \tilde{p}}{\partial z} h, \tag{3d}$$

$$v_z(z=-H)=0. \tag{3e}$$

(ii) *Small Froude number.* In calculating wave scattering by turbulence we take into account that the value of typical turbulent speed fluctuations U is usually small in comparison with the phase speed of gravity waves $v_p = (g/k)^{\frac{1}{2}}$ and so the Froude number is also small:

$$F = kU^2/g \ll 1. \tag{4}$$

(iii) *Quazi-static approximation.* We assume the turbulent vortex flow to be much slower than the wave motion and, therefore, if their space scales are the same, the ratio of turbulent frequency Ω_t to the wave frequency is small:

$$\Omega_t \ll \omega. \tag{5}$$

The condition (5) makes it possible to consider wave scattering by turbulence in the quasi-static approximation, i.e. to take into consideration the time dependence of turbulent velocity only in the final expressions. This condition is always valid if the Froude number of a flow is small enough: $F \ll 1$ for vortices with a distance scale l greater than the wavelength λ. Vortices with scales $l < \frac{1}{2}\lambda$ do not participate in resonant wave scattering.

(iv) *Subsurface flow under a rigid boundary.* The undisturbed flow is described by the system of equations

$$\frac{\partial U}{\partial t} + \nabla\left(\frac{U^2}{2}+\frac{P}{\rho}\right) = U \times [\nabla \times U], \tag{6a}$$

$$\nabla \cdot U = 0, \tag{6b}$$

$$\frac{\partial h}{\partial t} + (U_\perp \cdot \nabla_\perp) h = U_z(z=0) + \left[\frac{\partial U_z}{\partial z}\right]_{z=0} h, \tag{6c}$$

$$p(z=0) + \left[\frac{\partial P}{\partial z}\right]_{z=0} h = \rho g h, \tag{6d}$$

$$U_z(z=-H)=0. \tag{6e}$$

Using condition (1) we can neglect in (6) the terms containing time derivatives. In this approximation (6*a*) and (6*d*) lead to the following relations:

$$P \sim \rho U^2, \quad h \sim U^2/g, \quad k_t h \sim F. \tag{7}$$

Here $k_t = 2\pi/L_t$ is the characteristic size of vortices.

Also, it follows from (6*b*) that

$$\frac{\partial U_z}{\partial z} = -(\nabla_\perp \cdot U_\perp). \tag{8}$$

Substituting (8) into (6*c*) we find that

$$U_z(z=0) \sim k_t h U \sim FU \sim k_t U^3/g. \tag{9}$$

If the Froude number is small (in the limit $U \to 0$) it follows from (9) that $U_z(z=0) \ll U_\perp$ and we can use in this approximation the boundary condition:

$$U_z(z=0)=0. \tag{10}$$

Thus to a first approximation in the Froude number the background vortex flow can be considered as a flow near a rigid boundary at $z=0$.

Under the approximations given above we can neglect the right-hand sides of (3) and rewrite them in the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{\nabla}\left(\frac{\tilde{p}}{\rho}\right) = -(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{U}-(\boldsymbol{U}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}, \tag{11a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \tag{11b}$$

$$\frac{\partial \eta}{\partial t}-v_z(z=0) = -\boldsymbol{\nabla}_\perp\cdot(\boldsymbol{U}_\perp\eta), \tag{11c}$$

$$\tilde{p}(z=0)-\rho g\eta = 0, \tag{11d}$$

$$v_z(z=-H) = 0. \tag{11e}$$

If $U = 0$ the right-hand sides of (11) equal zero and we have a system of equations describing free propagation of small surface waves. Non-zero terms on the right-hand sides of (11) determine the influence of an arbitrary subsurface vortex flow (but taking into account the approximations (4) and (5) on a surface wave propagation).

To calculate this influence it is convenient to use Fourier transformation of (11). We define the Fourier amplitudes as

$$p_k = \int \tilde{p}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^2r, \tag{12a}$$

$$\boldsymbol{v}_{zk} = \int \boldsymbol{v}(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^2r, \tag{12b}$$

$$\eta_k = \int \eta(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^2r, \tag{12c}$$

where $\boldsymbol{r} = (x, y)$ is a two-dimensional vector, and index $\boldsymbol{k}$ designates the amplitudes of $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$ harmonics. It can be found from (11) that

$$\frac{\mathrm{d}^2\eta_k}{\mathrm{d}t^2} = -\left[\frac{\partial}{\partial z}\left(\frac{\tilde{p}_k}{\rho}\right)+2(\boldsymbol{U}_\perp\cdot\boldsymbol{\nabla}_\perp v_z)_k\right]_{z=0}. \tag{13}$$

To find $\tilde{p}_k/\rho$ we use (11*a*), which leads to the equation

$$\frac{1}{\rho}\left(\frac{\mathrm{d}^2p_k}{\mathrm{d}z^2}-k^2p_k\right) = -\Pi_k, \tag{14}$$

where

$$\Pi = \boldsymbol{\nabla}_\perp\cdot\left[(\boldsymbol{v}\cdot\boldsymbol{\nabla}_\perp)\,\boldsymbol{U}+(\boldsymbol{U}\cdot\boldsymbol{\nabla}_\perp)\,\boldsymbol{v}+v_z\frac{\partial \boldsymbol{U}}{\partial z}+U_z\frac{\partial \boldsymbol{v}}{\partial z}\right]$$
$$+\frac{\partial}{\partial z}\left[(\boldsymbol{v}\cdot\boldsymbol{\nabla}_\perp)\,U_z+(\boldsymbol{U}\cdot\boldsymbol{\nabla}_\perp)\,v_z+v_z\frac{\partial U_z}{\partial z}+U_z\frac{\partial v_z}{\partial z}\right]. \tag{15}$$

Note here that we use the approximation of small Froude number and so we may neglect all terms containing higher powers of the vortex velocity U in equations for surface waves. Taking into account boundary conditions at $z = -H$, we find from (14) to a first approximation:

$$\frac{p_k}{\rho} = A\cosh[k(z+H)]-\frac{1}{k}\int_{-H}^{z}\Pi_k\sinh[k(z-z')]\,\mathrm{d}z'. \tag{16}$$

Now we substitute (15) into (16) and use the continuity equations for the perturbation velocity $\boldsymbol{v}$ and for the background velocity U ((11*b*) and (6*b*)). After integrating by parts and taking into account the boundary conditions at $z=-H$ we find

$$\frac{p_k}{\rho} = A\cosh[k(z+H)] - \frac{1}{k}\int_{-H}^{z}\sinh[k(z-z')][\nabla_\perp((\boldsymbol{v}\cdot\nabla_\perp)\boldsymbol{U}+(\boldsymbol{U}\cdot\nabla_\perp)\boldsymbol{v} + \boldsymbol{U}(\nabla_\perp\cdot\boldsymbol{v})+\boldsymbol{v}(\nabla_\perp\cdot\boldsymbol{U}))]_k\,\mathrm{d}z' + 2\int_{-H}^{z}\cosh[(k(z-z')][(\boldsymbol{v}\cdot\nabla_\perp)U_z+(\boldsymbol{U}\cdot\nabla_\perp)v_z]_k\,\mathrm{d}z'. \tag{17}$$

It follows from (17) that

$$\left[\frac{\partial}{\partial z}\left(\frac{\tilde{p}_k}{\rho}\right)\right]_{z=0} = k\tanh(kH)\left(\frac{\tilde{p}_k}{\rho}\right)_{z=0} - \int_{-H}^{0}\frac{\cosh[k(z+H)]}{\cosh(kH)}[\nabla_\perp((\boldsymbol{v}\cdot\nabla_\perp)\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla_\perp)\boldsymbol{v}+\boldsymbol{U}(\nabla_\perp\cdot\boldsymbol{v})+\boldsymbol{v}(\nabla_\perp\cdot\boldsymbol{U}))]_k\,\mathrm{d}z - 2k\int_{-H}^{0}\frac{\sinh[k(z+H)]}{\cosh(kH)}[(\boldsymbol{v}\cdot\nabla_\perp)U_z+(\boldsymbol{U}\cdot\nabla_\perp)v_z]_k\,\mathrm{d}z - 2[((\boldsymbol{v}\cdot\nabla_\perp)U_z+(\boldsymbol{U}\cdot\nabla_\perp)v_z)_k]_{z=0}. \tag{18}$$

The condition (9) allows us to omit the last term $[\{(\boldsymbol{U}\cdot\nabla_\perp)v_z\}_k]_{z=0}$ in (18). Substituting (18) into (13) and using (11*d*) to express $(\tilde{p}_k/\rho)_{z=0}$ in terms of η_k we arrive at the equation

$$\frac{\mathrm{d}^2\eta_k}{\mathrm{d}t^2}+\omega_k^2\cdot\eta_k = \int\frac{\mathrm{d}^2k'}{(2\pi)^2}\int_{-H}^{0}\mathrm{d}z\frac{-2}{\cosh(kH)}[\cosh[k(z+H)](\boldsymbol{k}\cdot\boldsymbol{U}_{\perp(k-k')})(\boldsymbol{k}\cdot\boldsymbol{v}_{\perp k'}) + \mathrm{i}k\sinh[k(z+H)][((\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}_{\perp k'})U_{z(k-k')} + (\boldsymbol{k}'\cdot\boldsymbol{U}_{\perp(k-k')})v_{zk'}]], \tag{19}$$

where $\omega_k^2 = gk\tanh(kH)$.

Confining ourselves to the first approximation we should take the velocity amplitudes $\boldsymbol{v}_k$ to be of zero order in the vortex velocity U, corresponding to a free plane potential wave:

$$v_{zk} = \frac{\mathrm{d}\eta_k}{\mathrm{d}t}\frac{\sinh[k(z+H)]}{\sinh(kH)}, \quad \boldsymbol{v}_{\perp k} = \frac{\mathrm{i}\boldsymbol{k}}{k^2}\frac{\partial v_{zk}}{\partial z}. \tag{20}$$

Substituting them into the right-hand side of (19) and then taking into account the incompressibility condition

$$\frac{\partial}{\partial z}U_{z(k-k')}+\mathrm{i}((\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{U}_{\perp(k-k')}) = 0, \tag{21}$$

which is found from (8), we have from (19)

$$\frac{\mathrm{d}^2\eta_k}{\mathrm{d}t^2}+\omega_k^2\cdot\eta_k = \int\frac{\mathrm{d}^2k'}{(2\pi)^2}\int_{-H}^{0}\mathrm{d}z(-\mathrm{i}k)\frac{\mathrm{d}\eta_{k'}}{\mathrm{d}t} \times[((\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{U}_{\perp(k-k')})A_1(z,\boldsymbol{k},\boldsymbol{k}')+\mathrm{i}U_{z(k-k')}A_2(z,\boldsymbol{k},\boldsymbol{k}')], \tag{22}$$

where

$$A_1(z,\boldsymbol{k},\boldsymbol{k}') = \frac{1}{2\sinh(k'H)\cosh(kH)} \times \left[\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{kk'}\right)\cosh[(k+k')(z+H)] - \left(1-\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{kk'}\right)\cosh[(k-k')(z+H)]\right], \quad (23)$$

$$A_2(z,\boldsymbol{k},\boldsymbol{k}') = \frac{1}{2\sinh(k'H)\cosh(kH)}\left[(k+k')\left(1-\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{kk'}\right)\sinh[(k-k')(z+H)] - (k-k')\left(1+\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{kk'}\right)\sinh[(k+k')(z+H)]\right]. \quad (24)$$

The integro-differential equation (22) derived in the first-order approximation to $F^{\frac{1}{2}} = kU/\omega_k$ is convenient for the investigation of surface wave scattering at a quasi-static vortex flow employing a perturbation method.

3. The scattering amplitude for surface gravity waves

Now let us calculate in the Born approximation the scattering amplitude for a wave in a vortex flow in a given volume. We can calculate the scattering of waves with various scales (wind waves, swells, tides, tsunamis, etc.) by any arbitrary distributed dynamical vortices in an ocean of finite depth. Here we confine ourselves to the investigation of some special cases. To make the first Born approximation we substitute into the right-hand side of (22) the expression corresponding to a monochromatic plane incident wave:

$$\eta_{k'} = (2\pi)^2\,\eta_0 \exp(-\mathrm{i}\omega t)\,\delta(\boldsymbol{k}'-\boldsymbol{k}_0), \quad (25)$$

where $\omega^2 = gk_0$. It implies that the incident wave is scattered only once. Thus we have the first-order scattered field amplitude:

$$\frac{\eta_k^{(1)}}{\eta_0} = \frac{\omega k}{\omega^2-\omega_k^2}\int_{-H}^{0} \mathrm{d}z[((\boldsymbol{k}+\boldsymbol{k}_0)\cdot \boldsymbol{U}_{\perp(k-k_0)})\,A_1(z,\boldsymbol{k},\boldsymbol{k}_0) + \mathrm{i}U_{z(k-k_0)}\,A_2(z,\boldsymbol{k},\boldsymbol{k}_0)]. \quad (26)$$

The space distribution of gravity waves is determined with the poles $\omega^2 = \omega_k^2$ in (26). The integration path in the complex plane of k_x for the reverse Fourier transform is defined by causality and must turn round the pole $\omega_k = \omega$ in a direction opposite to clockwise. Integrating along this path we have from (26)

$$\frac{\eta^{(1)}}{\eta_0} = \frac{-\mathrm{i}}{2\pi}\int \frac{\mathrm{d}^2k_y}{(2\pi)^2}\mathrm{e}^{\mathrm{i}k_{sx}x+\mathrm{i}k_y y}\frac{k^2}{2(\mathrm{d}\omega/\mathrm{d}k)\,k_{sx}}\int_{-H}^{0}\mathrm{d}z[((\boldsymbol{k}_s+\boldsymbol{k}_0)\cdot \boldsymbol{U}_{\perp(k_s-k_0)})\,A_1(z,\boldsymbol{k}_s,\boldsymbol{k}_0)]. \quad (27)$$

Here $\boldsymbol{k}_s = (k_{sx}, k_y)$, $k_{sx}(k_y) = (k^2-k_y^2)^{\frac{1}{2}}$, and $k = k_0$. We have taken into account that for $\boldsymbol{k}_s = \boldsymbol{k}_0$, $A_2 = 0$.

Using the stationary phase method one can obtain from (27) an asymptotic form for a scattered wave in the far field when $kr \gg 1$:

$$\frac{\eta^{(1)}}{\eta_0} = f(\boldsymbol{k}_s,\boldsymbol{k}_0)\frac{\exp(\mathrm{i}k_s r)}{r^{\frac{1}{2}}}. \quad (28)$$

Here

$$f(\boldsymbol{k}_s,\boldsymbol{k}_0) = \frac{\mathrm{i}k_0}{2(\mathrm{d}\omega/\mathrm{d}k)}\left[\frac{k_0}{2\pi\mathrm{i}}\right]^{\frac{1}{2}}\int_{-H}^{0}\mathrm{d}z[((\boldsymbol{k}_s+\boldsymbol{k}_0)\cdot \boldsymbol{U}_{\perp(k_s-k_0)})\,A_1(z,\boldsymbol{k}_s,\boldsymbol{k}_0)] \quad (29)$$

is the scattering amplitude. The scattered field $\eta_k^{(1)}$ depends upon the radius vector in the horizontal plane $\boldsymbol{r} = (x, y)$. And the scattering vector $\boldsymbol{k}_s$ is directed along the radius-vector $\boldsymbol{k}_s = k_s \boldsymbol{r}/r$.

Now we can find $(\boldsymbol{k}_s + \boldsymbol{k}_0)\cdot \boldsymbol{U}_{\perp(\boldsymbol{k}_s-\boldsymbol{k}_0)}$, which the integrand in (29) is proportional to. It is convenient to expand the horizontal turbulent velocity $\boldsymbol{U}_\perp$ into two components: a vortical velocity that is determined by the vertical component Ω_z of vorticity $\boldsymbol{\Omega} = \operatorname{rot} \boldsymbol{U}$, and a potential component that is determined by U_z:

$$\boldsymbol{U}_{\perp q} = \frac{[\boldsymbol{n}_z \times \boldsymbol{q}]}{\mathrm{i}q^2}\Omega_z(\boldsymbol{q}) + \frac{\mathrm{i}\boldsymbol{q}}{q^2}\frac{\partial U_{zq}}{\partial z}. \tag{30}$$

Here $\boldsymbol{q} = \boldsymbol{k}_s - \boldsymbol{k}_0$, $q = 2k_0 \sin(\frac{1}{2}\theta)$, and $\boldsymbol{n}_z$ is the vertical unit vector. This expansion leads to an equation

$$(\boldsymbol{k}_s + \boldsymbol{k}_0)\,\boldsymbol{U}_{\perp q} = \frac{2([\boldsymbol{k}_s \times \boldsymbol{k}_0]\cdot \boldsymbol{n}_z)}{\mathrm{i}q^2}\Omega_z(\boldsymbol{q}) + \mathrm{i}\frac{k_s^2 - k_0^2}{q^2}\frac{\partial U_{zq}}{\partial z}. \tag{31}$$

One can see from (31) and (29) that when $k_s = k_0$ the scattering amplitude $f(\boldsymbol{k}_s, \boldsymbol{k}_0)$ is determined only by the vertical component of vorticity. This circumstance seems to be rather useful from the experimental point of view because exploration of Ω_z requires measuring only horizontal velocity components.

Substituting (31) and (23), where $k' = k_0$ and $k = k_s$, into (29), we find

$$f(\boldsymbol{k}_s, \boldsymbol{k}_0) = \frac{k_0}{2(\mathrm{d}\omega/\mathrm{d}k)}\left[\frac{k_0}{2\pi \mathrm{i}}\right]^{\frac{1}{2}} \frac{2([\boldsymbol{k}_s \times \boldsymbol{k}_0]\cdot \boldsymbol{n}_z)}{(\boldsymbol{k}_s - \boldsymbol{k}_0)^2}$$
$$\times \frac{1}{\sinh(2k_0 H)}\int_{-H}^{0}\left[\left(1 + \frac{\boldsymbol{k}_s\cdot \boldsymbol{k}_0}{k_s k_0}\right)\cosh[2k(z+H)] - \left(1 - \frac{\boldsymbol{k}_s\cdot \boldsymbol{k}_0}{k_s k_0}\right)\right]\Omega_{z(\boldsymbol{k}_s-\boldsymbol{k}_0)}\,\mathrm{d}z. \tag{32}$$

For shallow water waves ($H \to 0$) the scattering amplitude (32) has the same form as for sound waves scattered at two-dimensional vortices (see Fabrikant 1983):

$$f(\boldsymbol{k}_s, \boldsymbol{k}_0) = \frac{1}{2(gH)^{\frac{1}{2}}}\left[\frac{k_0}{2\pi \mathrm{i}}\right]^{\frac{1}{2}} \frac{\boldsymbol{k}_s\cdot \boldsymbol{k}_0}{k_s k_0}\frac{2([\boldsymbol{k}_s \times \boldsymbol{k}_0]\cdot \boldsymbol{n}_z)}{(\boldsymbol{k}_s - \boldsymbol{k}_0)^2}\bar{\Omega}_{z(\boldsymbol{k}_s-\boldsymbol{k}_0)}, \tag{33}$$

where

$$\bar{\Omega}_{z(\boldsymbol{k}_s-\boldsymbol{k}_0)} = \frac{1}{H}\int_{-H}^{0}\mathrm{d}z\,\Omega_{z(\boldsymbol{k}_s-\boldsymbol{k}_0)} \tag{34}$$

is simply the vertically averaged vorticity. This result seems to be quite evident due to the analogy of shallow-water approximation with two-dimensional acoustics (Landau & Lifshits 1987).

For deep-water waves ($H \to \infty$) we have from (32)

$$f(\boldsymbol{k}_s, \boldsymbol{k}_0) = \frac{k}{2(2\pi \mathrm{i}g)^{\frac{1}{2}}}\left(1 + \frac{\boldsymbol{k}_s\cdot \boldsymbol{k}_0}{k_s k_0}\right)\frac{2([\boldsymbol{k}_s \times \boldsymbol{k}_0]\cdot \boldsymbol{n}_z)}{(\boldsymbol{k}_s - \boldsymbol{k}_0)^2} I_{(\boldsymbol{k}_s-\boldsymbol{k}_0)}, \tag{35}$$

where

$$I_{(\boldsymbol{k}_s-\boldsymbol{k}_0)} = 2k_0\int_{-H}^{0}\mathrm{d}z\,\Omega_{z(\boldsymbol{k}_s-\boldsymbol{k}_0)}\,\mathrm{e}^{2k_0 z}, \tag{36}$$

$$I(\boldsymbol{r}) = 2k_0\int_{-H}^{0}\mathrm{d}z\,\Omega_z(\boldsymbol{r})\cdot \mathrm{e}^{2k_0 z}. \tag{37}$$

We can see from (35) and (37) that only subsurface distribution of the vertical vorticity up to the depth $|z| \leqslant l_k = k_0^{-1}$ determines surface wave scattering in deep water.

Below, our discussion is confined to the deep-water waves, taking into account the possibility of a generalization for a finite depth.

4. Scattering at localized vortices

First, let us consider some general properties of the scattering amplitude. We can introduce the scattering angle θ by the relation

$$\frac{\boldsymbol{k}_s \cdot \boldsymbol{k}_0}{k_s k_0} = \cos\theta \tag{38}$$

and rewrite (35) in the following form:

$$f(\boldsymbol{k}_s, \boldsymbol{k}_0) = \frac{k}{2(2\pi \mathrm{i} g)^{\frac{1}{2}}} \cot(\tfrac{1}{2}\theta)(1+\cos\theta)\, I_{(\boldsymbol{k}_s-\boldsymbol{k}_0)}. \tag{39}$$

The formula (39) is very convenient for investigation of scattering at simple localized vortex flows with given vorticity distribution $\Omega_z(\boldsymbol{r}, z)$.

One can see from (39) that there is no backward scattering: for $\theta = \pi$ we have $f(-\boldsymbol{k}_0, \boldsymbol{k}_0) = 0$. The scattering at small angles ($\theta = 0$) depends on the value of circulation around the vortex $2\pi\kappa$ averaged along the scattering layer, where

$$\begin{aligned} 2\pi\kappa = I_{(\boldsymbol{k}_s-\boldsymbol{k}_0=0)} &= 2k_0 \int \mathrm{d}^2 r \int_{-H}^{0} \mathrm{d}z\, \Omega_z(r, z)\, \mathrm{e}^{2k_0 z} \\ &= 2k_0 \int_{-H}^{0} \left[\oint U_\perp(\boldsymbol{r}, z)\, \mathrm{d}\boldsymbol{l} \right] \mathrm{e}^{2k_0 z}\, \mathrm{d}z. \end{aligned} \tag{40}$$

If $I_{(\boldsymbol{k}_s-\boldsymbol{k}_0=0)} \neq 0$ the scattering amplitude has a singularity (a pole) at $\theta = 0$. This pole is due to a slow decrease ($\propto r^{-1}$) of the velocity $U_\perp$ when $r \to \infty$ so that even far-distant rays are refracted in this kind of flow field.

The vorticity field may be expanded into a multipoles series

$$I_{(\boldsymbol{k}_s-\boldsymbol{k}_0)} = I_{(\boldsymbol{k}_s-\boldsymbol{k}_0=0)} + (\boldsymbol{k}_s - \boldsymbol{k}_0)\cdot \boldsymbol{P} + \ldots, \tag{41}$$

where the vector

$$\boldsymbol{P} = \int \boldsymbol{r} \cdot I(\boldsymbol{r})\, \mathrm{d}^2 r \tag{42}$$

is the dipole moment of vorticity in the scattering layer. In the long-wave approximation when the typical horizontal size of a vortex is small compared to the wavelength ($k_0 L \ll 1$) we may confine ourself to the first non-vanishing term of the series (41). If the total flow vorticity $I_{(\boldsymbol{k}_s-\boldsymbol{k}_0=0)} \neq 0$, the vortex flow can be replaced in the first approximation by a point vortex with circulation $2\pi\kappa$. Then the scattering amplitude has the form

$$f = [2\pi/\mathrm{i}g]^{\frac{1}{2}} \tfrac{1}{2}\kappa k_0 \cot(\tfrac{1}{2}\theta)(1+\cos\theta). \tag{43}$$

Therefore, the scattering of long waves is determined by the vortex oscillations as a whole and does not depend on the vortex inner structure. Note that (43) is an accurate formula for the scattering amplitude at a vertical vortex line whose length is large in comparison with the wavelength.

For vortex flows without circulation when $I_{(k_s-k_0=0)}=0$ the scattering amplitude has the form

$$f(\boldsymbol{k}_s,\boldsymbol{k}_0)=\frac{(2\pi i k_0)^{\frac{3}{2}}}{i\omega}k_0(1+\cos\theta)\,[P_y(\cos\theta+\cos\theta_0)+P_x(\sin\theta+\sin\theta_0)], \tag{44}$$

where θ_0 is the direction angle of the incident wave. The function $f(\theta)$ is finite in this case for all angles θ.

We supposed above an arbitrary flow vorticity distribution $I(\boldsymbol{r})$. The only condition on this function is its fast decreasing when $|\boldsymbol{r}|\to\infty$ so that Fourier transformation leads to the finite function $I_{(k_s-k_0)}$.

Now we consider an axially symmetrical vortex with the Fourier-transformed vorticity depending only upon $k=|\boldsymbol{k}|$. To calculate $I_{(k_s-k_0)}$ we use polar coordinates with the polar axis directed along the vector $\boldsymbol{k}$. Then we have

$$I_k=\int_0^\infty\int_0^{2\pi} I(r)\,e^{ikr\cos\psi}r\,d\psi\,dr=2\pi\int_0^\infty rJ_0(kr)\,I(r)\,dr. \tag{45}$$

This integral is finite if $I(r)$ decreases like r^{-s} (where $s>\frac{3}{2}$) when $r\to\infty$.

For the Oseen vortex with vorticity $I(r)=2B\exp(-r^2/R_0^2)$ we have from (45) $I_k=2\pi BR_0^2\exp(-k^2R_0^2)$ and the scattering amplitude (39) has the following form:

$$f=\frac{\pi BR_0^2k_0}{(2\pi ig)^{\frac{1}{2}}}\cot(\tfrac{1}{2}\theta)(1+\cos\theta)\exp[-4k_0^2R_0^2\sin^2(\tfrac{1}{2}\theta)]. \tag{46}$$

Some useful results can be found for surface wave scattering using a point vortices model for the averaged undisturbed flow vorticity $I(\boldsymbol{r})$. Approximation of a fluid flow by a system of point vortices has been widely used before (see, for example, Batchelor 1970). It would be quite natural to make an analogy (kinematical, at least) between point vortices in fluid dynamics and point charges in electrodynamics. But there is a significant difference between these two. Wave scattering in a point vortex flow is determined not only by the vortex core but also by the refraction in the flow velocity field that stretches far away from the core. As has been noted before, this leads to a peculiarity in the scattering amplitude for small angles. On the other hand, the point charge Coulon field does not influence the electromagnetic wave scattering due to the linear superposition principle in electrodynamics.

Now we calculate the scattering amplitude for some simple systems of point vortices. In particular, for a 'double vortex' – a pair of point vortices with equal circulations κ, that rotates around a centre between them with the frequency $\Omega_0=2\kappa/d^2$ (d is the distance between vortices) – we have the formula for coordinates of these vortices:

$$\boldsymbol{r}_i=(\tfrac{1}{2}d\cos(\Omega_0 t+i\pi),\tfrac{1}{2}d\sin(\Omega_0 t+i\pi)),\quad i=1,2. \tag{47}$$

The scattering amplitude has the form

$$\begin{aligned}f(\theta)&=i\kappa(2\pi i/g)^{\frac{1}{2}}k_0(1+\cos\theta)\cot(\tfrac{1}{2}\theta)\\&\quad\times\cos\{\tfrac{1}{2}k_0d[(\cos\theta-\cos\theta_0)\cos\Omega_0t+(\sin\theta-\sin\theta_0)\sin\Omega_0t]\}\\&=i\kappa(2\pi i/g)^{\frac{1}{2}}k_0(1+\cos\theta)\cot(\tfrac{1}{2}\theta)\\&\quad\times\sum_m(-1)^mJ_m(k_0d\sin\tfrac{1}{2}(\theta-\theta_0))\cos[m(\Omega_0t-\tfrac{1}{2}(\theta+\theta_0))],\end{aligned} \tag{48}$$

where θ_0 is the incident wave vector angle. We can see from (48) that a wave field scattered from this oscillated flow includes an infinite spectrum of combined harmonics with frequencies $\omega \pm m\Omega_0$ $(m = 0, 1, 2, \ldots)$.

For a vortex pair with opposite circulations that moves along the x-axis with the speed $V = \kappa/d$ we have

$$r_i = \{V_0 t, (-1)^i \tfrac{1}{2}d\}, \quad i = 1, 2, \tag{49}$$

and the scattering amplitude is

$$f(\theta) = -\kappa(-2\pi \mathrm{i}/g)^{\frac{1}{2}} k_0 (1 + \cos\theta) \cot(\tfrac{1}{2}\theta) \times \sin[\tfrac{1}{2}k_0 d(\sin\theta - \sin\theta_0)] \, \mathrm{e}^{\mathrm{i}k_0 V(\cos\theta - \cos\theta_0)t}. \tag{50}$$

In the frame of reference confined to the vortex pair and moving with the velocity V the flow is steady and the scattered wave frequency equals the incident wave frequency. For a moving observer the latter frequency is

$$\omega' = \omega\left(1 - \frac{k_0 V \cos\theta_0}{\omega}\right). \tag{51}$$

Therefore the scattered wave frequency in a fixed frame of reference differs from the incident wave frequency and the difference is the Doppler frequency shift:

$$\Delta\omega = \omega \frac{k_0 V}{\omega} \cos\theta \tag{52}$$

calculated in the first approximation (compare to Fabrikant 1983).

5. Scattering by homogeneous turbulence

Now we investigate a turbulent background flow that has no mean vertical vorticity:

$$\langle \Omega_z \rangle = 0. \tag{53}$$

Let us consider a turbulent volume confined by an area of size L in the horizontal plane. Suppose the turbulence statistics are horizontally homogeneous inside that area and so the correlation function may be written as

$$\langle \Omega_z(r', z') \Omega_z(r'', z'') \rangle = Q(r' - r'', z', z''). \tag{54}$$

The latter condition may be approximately satisfied for $|r' - r''| \leqslant L_t \ll L$ if the area size L is large enough and outside some boundary layer of width equal to the correlation length L_t.

It follows from (39), (37) and (54) that the differential scattering cross-section for deep water waves is

$$\mathrm{d}\sigma = \langle |f(\boldsymbol{k}_s, \boldsymbol{k}_0)|^2 \rangle \, \mathrm{d}\theta = S \frac{k_0^4}{2\pi g} (1 + \cos\theta)^2 \cot^2(\tfrac{1}{2}\theta) \int_{-\infty}^{0} \mathrm{d}z' \int_{-\infty}^{0} dz'' Q(\boldsymbol{q}, z', z'') \, \mathrm{e}^{2k_0(z'+z'')} \, \mathrm{d}\theta. \tag{55}$$

Here S denotes the scattering area, and

$$Q(\boldsymbol{q}, z', z'') = \int Q(\boldsymbol{r}, z', z'') \exp(-\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}) \, \mathrm{d}^2 r. \tag{56}$$

Two simple models of homogeneous subsurface turbulence may be considered. The first model is the two-dimensional turbulent flow where vorticity does not depend on depth. The vorticity correlation function $Q(\boldsymbol{q})$ in this approximation is a function only of the horizontal wave vector $\boldsymbol{q}$ and so we have from (55)

$$\mathrm{d}\sigma = \frac{S}{4}\frac{k_0^2}{2\pi g}(1+\cos\theta)^2\cot^2(\tfrac{1}{2}\theta)\,Q(\boldsymbol{q})\,\mathrm{d}\theta. \tag{57}$$

Another simple model is the three-dimensional homogeneous turbulence with the correlation function

$$Q(\boldsymbol{q}, z', z'') = Q(\boldsymbol{q}, z'-z''). \tag{58}$$

In that case we can use new variables

$$z_+ = z'+z'', \quad z_- = z'-z'' \tag{59}$$

and find from (55)

$$\mathrm{d}\sigma = \frac{S}{4}\frac{k_0^3}{2\pi g}(1+\cos\theta)^2\cot^2(\tfrac{1}{2}\theta)\int_{-\infty}^{+\infty} Q(\boldsymbol{q}, z_-)\,\mathrm{d}z_-\,\mathrm{d}\theta. \tag{60}$$

For further analyses we must know a vorticity spectrum $Q(\boldsymbol{q})$. In particular, the well-known Kolmogorov spectrum for homogeneous and isotropic turbulence may be used (Landau & Lifshits 1987). The Kolmogorov spectrum of vorticity may be written in the following form:

$$Q(\boldsymbol{k}) = Ck^{-\frac{5}{3}}, \tag{61}$$

where $k = (q_x^2+q_y^2+q_z^2)^{\frac{1}{2}}$ (three-dimensional spectrum).

It is evident however that the total scattering cross-section

$$\sigma = \int \mathrm{d}\sigma \tag{62}$$

diverges at small scattering angles for the Kolmogorov spectrum (61), where $k = 2k_0\sin(\frac{1}{2}\theta)\rightarrow 0$. And so do some other characteristics of scattering considered below. All these scattering characteristics are determined by the large-scale spectrum components that are usually anisotropic and inhomogeneous and cannot be described by the Kolmogorov universal law. Therefore we should use a particular turbulence spectrum limited to large scales.

6. Surface wave scattering by the turbulence of a boundary layer

One of the most important types of turbulence in the upper ocean influencing surface waves is probably the drift flow turbulence. Owing to the boundary condition (10), derived in the first approximation on the Froude number, a subsurface turbulent flow may be considered as a flow near a rigid boundary at $z = 0$. This permits us to use the model of a logarithmic boundary layer (LBL) near a solid surface.

In the self-similar region of a turbulent drift flow when $z \ll L_t$ the correlation function depends on the dimensionless variables $\xi = \frac{1}{2}q(z'+z'')$ and $\chi = z''/z'$. Note that the exponent in (55) limits the integration area by the condition: $\xi \leqslant q/k_0 = 2\sin(\frac{1}{2}\theta) \leqslant 1$. Kader (1984) proved that under the condition $\xi \ll 1$ the correlation functions do not depend on ξ. Using this fact and the absence of vorticity correlation at scales $r \gg L_t$, we approximate the function $Q(\boldsymbol{q}, z', z'')$ by the expression

$$Q(\boldsymbol{q}, z', z'') = v_*^2\,\Phi(\boldsymbol{n}, \chi)\,H(q-q_t). \tag{63}$$

Here $\boldsymbol{n} = \boldsymbol{q}/q$, $q_t = 2\pi/L_t$, v_* is the LBL dynamic speed, $\Phi(\boldsymbol{n}, \chi)$ is a dimensionless function ($0 \leqslant \Phi \leqslant 1$) and

$$H(q) = \begin{cases} 0 & \text{if} \quad q \leqslant 0 \\ 1 & \text{if} \quad q > 0 \end{cases}$$

is the Heaviside step function. Then employing the approximation (63), from (55) we have

$$\langle \mathrm{d}\sigma \rangle = S \frac{k_0^2 v_*^2}{4\pi g} G(\boldsymbol{n}) (1 + \cos\theta)^2 \cot^2(\tfrac{1}{2}\theta)\, \mathrm{d}\theta, \tag{64}$$

where $\theta > 2\arcsin(q_t/2k_0)$ and

$$G(\boldsymbol{n}) = \int_0^{-\infty} \frac{\Phi(\boldsymbol{n}, \chi)}{(1+\chi)^2} \mathrm{d}\chi. \tag{65}$$

It should be noted that, if $q \sim q_t$, the LBL spectrum approximation (63) is rough and therefore in an area $\theta \sim q_t/k_0 \sim 1$ the expression (64) is suitable only for qualitative estimations. At the same time, if $q_t/k_0 \ll \theta \ll 1$, (64) gives the right asymptotic form for the scattering cross-section:

$$\frac{\langle \mathrm{d}\sigma \rangle}{S} = \frac{2k_0^2 v_*^2}{\pi g} \frac{G(\boldsymbol{n})}{\theta^2} \mathrm{d}\theta. \tag{66}$$

The scattering cross-section, although calculated in the single scattering approximation, may be used for investigation of some multiple scattering effects such as damping of the coherent field component $\langle \eta_k \rangle$ and transformation of wave angular distribution. The Bourret approximation, which implies that all higher terms except the first one in the perturbation expansion of the mass operator in Dyson equation are neglected (Rytov, Kravtsov & Tatarsky 1978), may be used if scattering fluctuations are small enough. It should be emphasized that use of that approximation is not based on a specific features of the given Green function and so does not depend on features of the scattering medium. For surface wave scattering problems a Froude number may be considered as a small parameter.

It may be proved under the Bourret approximation that the coherent field component decreases exponentially along the propagation distance and the space damping rate $\bar{\nu}$ for the coherent component (the extinction coefficient) is proportional to the total cross-section (per unit area) (Rytov *et al.* 1978)

$$\bar{\nu} = \frac{1}{2} \int \frac{\langle \mathrm{d}\sigma \rangle}{S} = \frac{k_0^2 v_*^2}{16\pi g} \int_{2\arcsin(q_t/2k_0)}^{\pi} G(\boldsymbol{n}) (1 + \cos\theta)^2 \cot^2(\tfrac{1}{2}\theta)\, \mathrm{d}\theta. \tag{67}$$

For short gravity waves ($k_0 L_t \gg 1$) the main contribution to the integral (67) is made by small angles $\theta \ll 1$. As a scattering angle corresponding to a turbulent scale L is of the order $\theta_L \sim (k_0 L)^{-1}$, so the damping of the coherent component is evidently determined mainly by large-scale turbulent vortices ($k_0 L \gg 1$). Taking into account the small value of the angle θ we can use instead of (67) an approximate formula

$$\bar{\nu} = G(\boldsymbol{n}_0) \frac{k_0^3 v_*^2 L_t}{2\pi^2 g}, \tag{68}$$

where $\boldsymbol{n}_0$ is a unit vector that is orthogonal to the wave vector $\boldsymbol{k}_0$. The dependence of $\bar{\nu}$ on the direction of the wave propagation is a consequence of an LBL anisotropy in the horizontal plane.

7. Surface wave angular spectrum evolution

Besides the damping of a coherent wave component, variations of an angular spectrum and correlation function due to the scattering are also interesting. This effect may compete with other known processes, for example, wind–wave interaction, nonlinear wave processes etc.

The spectrum evolution may be found for the case of practical importance when $k_0 L_t \gg 1$ and scattering is small at a distance L_t. In that case the 'staircase' approximation may be used. The staircase approximation is based on the use of small turbulent fluctuations (small Froude number in our case) and large turbulent scale ($k_0 L_t \gg 1$) as well. A transfer equation may be derived under this approximation from a Bethe–Salpeter equation (Rytov *et al.* 1978). Under this assumption a typical form of a transfer equation for wave intensity $J(\boldsymbol{r}, \boldsymbol{n})$ may be written

$$\boldsymbol{n}\cdot\nabla J(\boldsymbol{r},\boldsymbol{n}) = \int \sigma(\boldsymbol{n},\boldsymbol{n}')\, J(\boldsymbol{r},\boldsymbol{n}')\, \mathrm{d}^2 n' - 2\bar{\nu} J(\boldsymbol{r},\boldsymbol{n}), \tag{69}$$

where $\boldsymbol{n} = (n_x, n_y)$, $\boldsymbol{r} = (x, y)$ and $\sigma(\boldsymbol{n}, \boldsymbol{n}')$ is the scattering cross-section in the Born approximation.

We may replace the integration variable in (69) on the transverse angle θ and then, using the small-angle approximation (66) for the scattering cross-section, we find from (69) for a wave propagating along x a transfer equation in the form

$$\frac{\partial J}{\partial x} = \frac{8\pi k_0^2 v_*^2}{g} G(\boldsymbol{n}_0) \int_{|\theta| > q_t/k_0} J(\theta - \tilde{\theta}) \frac{\mathrm{d}\tilde{\theta}}{\tilde{\theta}^2} - 2\bar{\nu} J. \tag{70}$$

Here we take into account that the minimum scattering angle θ_{min} is determined by the maximum scale of a turbulent spectrum $q_t = 2\pi/L_t$.

To find a solution of the transfer equation it is more convenient to use instead of (70) an equivalent equation for the transverse correlation function of the complex wave amplitude $a(x, y)$

$$K(R, x) \equiv \langle a(x,y)\, a^*(x, y+R)\rangle = \int J(\theta) \exp(\mathrm{i} k_0 \theta R)\, \mathrm{d}(k_0 \theta). \tag{71}$$

Using the Fourier transformation we find from (70)

$$\frac{\partial}{\partial x} K(R, x) = [W(R) - W(0)]\, K(R, x), \tag{72}$$

where

$$W(R) = \frac{16\pi k_0^3 v_*^2}{g} \frac{G(\boldsymbol{n}_0)}{q_t} [\cos q_t R + q_t R\, Si(q_t R) - \tfrac{1}{2}\pi q_t |R|]. \tag{73}$$

Equation (72) has the solution

$$K(R, x) = K(R, 0) \exp([W(R) - W(0)]\, x), \tag{74}$$

which leads to an asymptotic expression for the correlation function:

$$K(R, x) = K(R, 0) \exp\left(-\frac{8\pi^2 k_0^3 v_*^2}{g} G(\boldsymbol{n}_0)\, |R|\, x\right) \tag{75}$$

corresponding to a distance x where the transverse correlation scale R_c for a wave field becomes less than the external turbulence scale L_t. For a typical correlation scale R_c (that is determined on the level e^{-1}) in a plane wave we have from (75)

$$R_c = \frac{g}{8\pi^2 k_0^3 v_*^2 G(\boldsymbol{n}_0)\, x} = \frac{L_t}{4\pi^2 \bar{\nu} x}. \tag{76}$$

The condition $k_0^{-1} \ll R_c \ll L_t$ determines the range of distances where (75) is valid:

$$(4\pi^2\bar{\nu})^{-1} \ll x \ll kL_t(4\pi^2\bar{\nu})^{-1}. \tag{77}$$

It should be emphasized that the transverse correlation function in this range does not depend on the external turbulence scale and is determined only by LBL features, thus being universal for arbitrary turbulent drift flows.

The wave intensity angular spectrum that corresponds the correlation function (75) with the boundary condition $K(R, 0) = 1$ is

$$J(\theta) = \frac{8\pi^2 k_0^3 v_*^2\, x}{g} G(\boldsymbol{n}_0) \left[\frac{64\pi^2 k_0^6 v_*^4\, G^2(\boldsymbol{n}_0)}{g^2} x^2 + k_0^2 \theta^2\right]^{-1}. \tag{78}$$

The spectral width $\tilde{\theta}$ can be found from (78):

$$\tilde{\theta} = 8\pi^2 G(\boldsymbol{n}_0) \frac{k_0^2 v_*^2}{g} x = \frac{4\pi^2 \bar{\nu}}{k_0 L_t} x. \tag{79}$$

This expression coincides with the natural definition $\tilde{\theta} = (k_0 R_c)^{-1}$.

Thus the correlation scale R_c of a plane wave at a distance $x = \bar{L} \equiv (4\pi^2\bar{\nu})^{-1}$ (that also is a characteristic of a coherent component) is of the order of L_t and the angular spectrum width: $\tilde{\theta} = (k_0 L_t)^{-1}$. The angular spectrum becomes significantly wider ($\tilde{\theta} \sim 1$) at a distance

$$x \sim L_\theta = g[8\pi^2 k_0^2 v_*^2\, G(\boldsymbol{n}_0)]^{-1} = (k_0 L_t)\, \bar{L}. \tag{80}$$

8. Discussion

When $k_0 L_t \gg 1$ (which is valid in many important situations), it is possible to derive an explicit form of the extinction coefficient $\bar{\nu}$ and to calculate the correlation function as well as the wave intensity angular spectrum. In that case the damping of a mean wave field is determined by large vortices with the scales $l \sim L_t$. The corresponding expression (68) gives the dependence of the decrement $\bar{\nu}$ on drift flow parameters and on the wavelength.

The process of transverse correlation evolution and angular spectrum widening for a wave that was initially plane can be divided into three stages if $k_0 L_t \gg 1$. At distances $x \leqslant \bar{L}$ scattering mainly occurs at large vortices with scales $l \sim L_t$. Then the coherent component of the wave field is transformed into a random one and the transverse correlation scale decreases down to the value $\sim L_t$. At the second stage that is determined by the condition (77), the wave field is incoherent and the angular spectrum widening is determined by scattering at vortices with scales that are in the self-similarity range $\lambda \ll l \ll L_t$. As a result, the parameters ρ_c and $\tilde{\theta}$ do not depend on the drift flow depth and are determined only by the speed v_* and the wave propagation direction. The angular spectrum width can be measured by remote sensing methods, which makes it possible, in principle, to realize remote sensing of drift shear flows.

Finally, at $x > L_\theta$ a widening of the angular spectrum occurs. That stage cannot be described by the theory developed above.

It would be very interesting to make an experimental test of the theoretical results found here. We give here some estimates for typical oceanic conditions of multiple scattering effects in LBL turbulence that have been considered above. If a wave of wavelength $\lambda = 30$ m propagates in a drift flow with a speed $v_* = 1$ cm s^{-1} and a characteristic lengthscale $L_t = 40$ m, the self-similar dynamics of an angular spectrum sets in at distances $x \geqslant \bar{L} = 3$ km. The width of the angular spectrum is about 12° at that distance.

It should be noted however that under the condition of moderate or strong wind, various competing factors influence the gravity wave propagation. In the first place, they are the wave interaction with the atmospheric boundary layer and the nonlinear interaction with wind waves. Generally speaking, these effects are comparable with scattering by a subsurface turbulence. The effects of scattering considered above can manifest themselves in a net form under a light wind with a speed close to the threshold of wind wave generation. For instance, considering wind with the speed 2 m s^{-1} ($v_* = 0.2$ cm s^{-1}) which does not affect waves with wavelengths longer than 1 m (cf. Phillips 1977) we have the value $\bar{L} = 1$ km for a wave with $\lambda = 3$ m and $L_t = 5$ m.

9. Conclusion

The scattering amplitude (32) derived in the Born approximation allows us to investigate some general properties of surface gravity wave scattering at vortex flows and to find the scattering amplitude for some particular localized vortices such as point vortices, axially symmetrical vortices, monopole and dipole vortices. The use of the Born approximation under the condition of small Froude number seems to be a rather weak restriction for cases of practical importance. A very important result found in this approximation is that the scattering amplitude is determined by the vertical component of the vorticity field.

Most known vortex flows have a relatively simple vorticity distribution in space. At the same time the velocity field for those flows is as a rule more complicated. This originates in the well-known fact that the vorticity is a conserving value and is transferred by inviscid fluid particles without change (Lamb 1947). In particular, if a flow is steady its vorticity is constant along closed streamlines. That is why expressions for wave scattering characteristics for a vortex flow in terms of the vorticity of that flow are more convenient than use of a velocity field. These characteristics has been investigated here for some localized vortex flows, in particular, point vortices.

Scattering by turbulent vortices seems to be of more geophysical interest. The estimations presented above show that any contemporary ocean wave prediction model (see Hasselmann *et al.* 1988) must anyhow take into account the effect of wave scattering by subsurface turbulence. For horizontally homogeneous turbulence the scattering cross-section is proportional to the vorticity correlation function integrated along the subsurface layer of a wavelength depth. The scattering of gravity waves by drift flow turbulence has been investigated employing the self-similarity of correlation functions. This is possible only if the external turbulence scale L_t determined by the LBL depth is greater than the wavelength $\lambda = 2\pi/k_0$. If $k_0 L_t \sim 1$, the expressions (64), (67) and (73) are suitable for qualitative evaluations.

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